HIERARCHY METHODS FOR RANDOM VIBRATIONS OF ELASTIC STRINGS AND BEAMS

by

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1. *Introduction*

Traditionally the normal frequencies and normal modes of vibration have played a very important role in the solution of many non-homogeneous boundary value problems in mathematical physics and mechanics. These normal modes are the solutions to eigenvalue problems which have been derived using various assumptions to simplify the differential equations and boundary conditions under consideration. Recently [1, 2]** attention has been given to various vibration problems in which the assumptions concerning some physical properties have been relaxed by considering them to be represented by stochastic variables. In recognition of this fact, certain of the coefficients appearing in the equations are assumed to be random functions or random variables, thereby making the solution a random function. The object, then is to determine as much statistical information as possible concerning the solution.

The analysis undertaken here involves the determination of various moments of the eigenfunctions. There are two methods of doing this. One method is to solve for the eigenfunction as accurately as possible and then to determine the moments. The second method is to average the equation first and then to solve these for the moments. The first is an "honest" method, whereas the second is a "dishonest" method [3]. This terminology arises from the fact that the equation for one moment of the eigenfunction leads to an infinite hierarchy of equations, and to obtain a finite set of equations, certain unjustified assumptions must be made. One of the main advantages of this "dishonest" method is that no assumptions on the smallness of the stochastic variable need be made to obtain very good approximations of the various moments over a wide range of values of the stochastic variable.

Hierarchy equation techniques have been widely used in the study of random wave propagation problems [3, 4] and random initial value problems in general [5]. However, their application to eigenvalue problems is considerably complicated by the fact that in addition to the random eigenfunctions the eigenvalues appear as random variables. This causes no particular difficulties in the first approximation, it is only in the second and higher approximations that we find that non-trivial closure assumptions must be made.

The analysis of hierarchy techniques as applied to stochastic eigenvalue problems is carried out on an example problem for which exact results and "honest" approximations are available for comparison. In the next section the hierarchy is derived and a first approximation for the mean value of the eigenfunctions u (denoted by $\langle u \rangle$) is obtained by retaining only the

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first equation of the infinite hierarchy. In section three the closure assumptions needed to insure an improved estimate for $\langle u \rangle$ are discussed along with a possible solution to the resulting determinant eigenvalue problem. Section four contains a discussion of the hierarchy for the determination of higher moments of the eigenfunctions and finally in the last section we discuss possible closure assumptions for some other typical eigenvalue problems.

2. *The Hierarchy and First Approximation*

The random vibration problem that we will use for the application of hierarchy techniques is that of the vibration string held fixed at one end and with an elastic constraint at the other. Mathematically, the normal modes and frequencies of such a vibration will be given by the eigenfunctions and eigenvalues of the problem:

$$\begin{aligned} u''(x) + \lambda\, u(x) &= 0 \\ u(0) &= 0 \\ u'(1) + \epsilon\, u(1) &= 0, \end{aligned} \tag{2.1}$$

where ϵ is a constant depending on the elastic constraint at $x = 1$. In the following work we will assume ϵ is a random variable for which the probability density function is known. For purposes of comparison to approximate solutions we will use the normalization condition $u'(0) = 1$. It should be mentioned that the reasons for picking this particular example are that it is typical of eigenvalue problems with the random function appearing in the boundary conditions and the exact solution and perturbation series solution are easily obtainable for comparison.

In order to find an estimate of the first moment of $u(x)$ directly from the equations (2.1) we must consider the set of equations:

$$\begin{aligned} \langle u \rangle''(x) + \langle \lambda u \rangle(x) &= 0 \\ \langle u \rangle(0) &= 0 \\ \langle u \rangle'(1) + \langle \epsilon u \rangle(1) &= 0 \\ \langle u \rangle'(0) &= 1; \end{aligned} \tag{2.2}$$

which are obtained from (2.1) by taking the mean value of each equation. In arriving at (2.2) we have assumed that the integral over the probability space and the differentiation with respect to x can be interchanged. Since the integrations involved are over finite intervals for applications used here, this assumption would not be hard to satisfy. Similar assumptions will be used throughout this paper. Equations (2.2) contain the two second order moments: $\langle \lambda u \rangle(x)$ and $\langle \epsilon u \rangle(1)$. To obtain equations for these quantities, multiply the equations (2.1) by λ and ϵ, respectively, and average. We then obtain the two sets of equations:

$$\begin{aligned} \langle \lambda u \rangle''(x) + \langle \lambda^2 u \rangle(x) &= 0 \\ \langle \lambda u \rangle(0) &= 0 \\ \langle \lambda u \rangle'(1) + \langle \lambda \epsilon u \rangle(1) &= 0 \end{aligned} \tag{2.3}$$

and

$$\begin{aligned} \langle \epsilon u \rangle''(x) + \langle \lambda \epsilon u \rangle(x) &= 0 \\ \langle \epsilon u \rangle(0) &= 0 \\ \langle \epsilon u \rangle'(1) + \langle \epsilon^2 u \rangle(1) &= 0 \end{aligned} \tag{2.4}$$

These equations, in turn, contain the three third order moments: $\langle \lambda^2 u \rangle(x)$, $\langle \lambda \epsilon u \rangle(x)$, and $\langle \epsilon^2 u \rangle(1)$, which must be determined in the next set of

equations of the hierarchy. Equations (2.2) are the first member of the hierarchy, and equations (2.3) and (2.4) are the second member.

To close the hierarchy at the first member, we must in someway replace $\langle \lambda u \rangle$ and $\langle \epsilon u \rangle$ by quantities involving first order moments only. The simplest assumptions are to assume $\langle \lambda u \rangle = \langle \lambda \rangle \langle u \rangle$ and $\langle \epsilon u \rangle = \langle \epsilon \rangle \langle u \rangle$. These assumptions, while clearly not altogether accurate, do permit the determination of $\langle u \rangle$. Under these assumptions, (2.2) becomes

$$\begin{aligned} \langle u \rangle''(x) + \langle \lambda \rangle \langle u \rangle (x) &= 0 \\ \langle u \rangle (0) &= 0 \\ \langle u \rangle'(1) + \langle \epsilon \rangle \langle u \rangle (1) &= 0 \\ \langle u \rangle'(0) &= 1. \end{aligned} \tag{2.5}$$

The solution of (2.5) is easily obtained as

$$\langle u \rangle (x) = \langle \lambda \rangle^{-1/2} \sin \langle \lambda \rangle^{1/2} x, \tag{2.6}$$

where, for $\langle \epsilon \rangle \neq 0$,

$$\langle \lambda \rangle^{1/2} = -\langle \epsilon \rangle \tan \langle \lambda \rangle^{1/2}. \tag{2.7}$$

Thus, not only do we get an approximation to the mean value of u(x), but also an approximation to the mean value of λ.

α	EXACT	HIERARCHY	4 TERM PERT.	3 TERM PERT.	2 TERM PERT.
.5	2.935	2.943	2.935	2.933	2.967
1	3.346	3.373	3.347	3.332	3.467
2	4.029	4.116	4.044	3.927	4.467
4	5.012	5.239	5.238	4.306	6.467
8	6.172	6.607			
16	7.283	7.865			
32	8.178	8.754			
50	8.617	9.128			

TABLE I: A comparison of $\langle \lambda \rangle$ for ϵ uniformly distributed between 0 and α.

The validity of the above unjustified assumptions can be seen in Table I in a numerical comparison of $\langle \lambda \rangle$ as given by (2.7) to the exact value (computed numerically to four digits) and to the value given by a perturbation series approximation, an "honest" technique. For this comparison ϵ is assumed to be uniformly distributed between 0 and α. For the discussions and comparisons in this paper only the first eigenvalue and eigenfunctions will be considered. All results and techniques discussed here can be extended to higher eigenvalues and eigenfunctions. A similar comparison is made for $\langle u \rangle (x)$ as given by (2.6) in Table II. In this case ϵ is assumed uniformly distributed between 0 and 1.

The numerical results given in Tables I and II show several advantages of the hierarchy technique over perturbation techniques. A close look at both tables shows that a one member hierarchy gives approximations that are better than a two term perturbation series approximation for all values of α given and better than a three term perturbation series approximation for all α greater than two. Further, in general a one member hierarchy is no more difficult to use than possibly a two term perturbation or certainly a three term perturbation series. Thus, for the same or less work than with a perturbation series, better results may be obtained by using

x	EXACT	HIERARCHY	3 TERM PERT.	2 TERM PERT.
.0	.0	.0	.0	.0
.1	.0994	.0994	.0994	.0992
.2	.1956	.1955	.1956	.1954
.3	.2852	.2850	.2852	.2846
.4	.3653	.3650	.3654	.3639
.5	.4332	.4326	.4336	.4306
.6	.4867	.4857	.4874	.4821
.7	.5241	.5225	.5252	.5167
.8	.5440	.5417	.5459	.5329
.9	.5459	.5426	.5489	.5299
1.0	.5296	.5254	.5343	.5076

TABLE II. A comparison of $\langle u \rangle(x)$ for ϵ uniformly distributed between 0 and 1.

hierarchy techniques.

A second, and perhaps more important, result is that the approximations as given in (2.6) and (2.7) are good for all $\langle\epsilon\rangle$, whereas the finite perturbation series approximations get continuously worse for larger epsilon, no matter how many terms are kept. This result is typical of solutions arising from such dishonest techniques as those used here and can be partly accounted for by the fact that nowhere in the above work did we assume that ϵ or $\langle\epsilon\rangle$ had to be small.

Further justification of the assumptions used in (2.5) are obtained when the solutions as given by (2.6) and (2.7) are expanded in a power series in $\langle\epsilon\rangle$, yielding respectively

$$\langle u \rangle (x) = u_0(x) + u_1(x) \langle\epsilon\rangle + u_2(x) \langle\epsilon\rangle^2 + \ldots \tag{2.8}$$

and

$$\langle\lambda\rangle = \lambda_0 + \lambda_1 \langle\epsilon\rangle + \lambda_2 \langle\epsilon\rangle^2 + \ldots \tag{2.9}$$

The perturbation series solutions are

$$\langle u \rangle (x) = u_0(x) + u_1(x) \langle\epsilon\rangle + u_2(x) \langle\epsilon^2\rangle + \ldots \tag{2.10}$$

and

$$\langle\lambda\rangle = \lambda_0 + \lambda_1 \langle\epsilon\rangle + \lambda_2 \langle\epsilon^2\rangle + \ldots, \tag{2.11}$$

which, of course, are correct through the highest order terms kept. Thus, we see that (2.6) and (2.7) can be justified through order $\langle\epsilon\rangle$ terms, as the two sets of coefficients $u_i(x)$ and λ_i appearing in (2.10) and (2.11) are the same as those appearing in (2.8) and (2.9). This also explains, in part, why the results as given in Tables I and II are better than a two term perturbation series.

3. *The Second and Higher Members of the Hierarchy*

The next step in the method of hierarchy equations is to consider closing the infinite set of equations at the second member of the hierarchy. The second member for the eigenvalue problem under consideration consists of the two sets of equations (2.3) and (2.4). These two sets of equa-

tions contain the three third order moments $\langle \epsilon^2 u \rangle$, $\langle \lambda^2 u \rangle$, and $\langle \lambda \epsilon u \rangle$, which must be expressed in terms of the three lower order moments $\langle u \rangle$, $\langle \lambda u \rangle$, $\langle \epsilon u \rangle$. When this is done, the resulting equations along with (2.2) result in a consistent system for the estimates of $\langle u \rangle (x)$, $\langle \epsilon u \rangle (x)$, and $\langle \lambda u \rangle (x)$. The situation occurring here is much more complicated than for a stochastic initial value problem [5], which has only one third order moment in the second member.

To determine what assumptions, if any, will give improved results, we must analyse more carefully the assumptions made in the last section when solving the first member. At that time we made the assumptions

$$\langle \lambda u \rangle = \langle \lambda \rangle \langle u \rangle \tag{3.1}$$

and

$$\langle \epsilon u \rangle = \langle \epsilon \rangle \langle u \rangle. \tag{3.2}$$

If we use perturbation series for λ and $u(x)$ and calculate the above products, we find that both (3.1) and (3.2) are correct to $\langle \epsilon \rangle$ terms. This not only tells us why $\langle \lambda \rangle$ and $\langle u \rangle (x)$ as given in the last section were correct to $\langle \epsilon \rangle$ terms, but also tells us that in order to guarantee an improvement in the means of $u(x)$ and λ we must make assumptions on the third order moments that are correct to terms involving $\langle \epsilon^2 \rangle$.

By looking at the appropriate perturbation series, it is possible to show that the following equalities for the third order moments are valid through $\langle \epsilon^2 \rangle$ terms,

$$\langle \epsilon^2 u \rangle = \langle \epsilon^2 \rangle \langle u \rangle ; \tag{3.3}$$

$$\langle \lambda^2 u \rangle = \langle \lambda^2 \rangle \langle u \rangle + 2 \langle \lambda \rangle \langle \lambda u \rangle - 2 \langle \lambda \rangle^2 \langle u \rangle, \tag{3.4}$$

and

$$\langle \lambda \epsilon u \rangle = \langle \lambda \epsilon \rangle \langle u \rangle + \langle \lambda \rangle \langle \epsilon u \rangle - \langle \epsilon \rangle \langle \lambda u \rangle \tag{3.5}$$

Using (3.3) to (3.5) in (2.2) to (2.4), we obtain a system of three coupled boundary value problems. If we let $m(x) = \langle u \rangle (x)$, $n(x) = \langle \epsilon u \rangle (x)$ and $p(x) = \langle \lambda u \rangle (x)$, then we can write them as

$$\begin{aligned} m''(x) &= -p(x) \\ m(0) &= 0 \\ m'(1) &= -n(1) \end{aligned} \tag{3.6}$$

$$\begin{aligned} n''(x) + \langle \lambda \rangle n(x) &= \langle \epsilon \rangle p(x) - \langle \lambda \epsilon \rangle m(x) \\ n(0) &= 0 \\ n'(1) &= -\langle \epsilon^2 \rangle m(1) \end{aligned} \tag{3.7}$$

and

$$\begin{aligned} p''(x) + 2\langle \lambda \rangle p(x) &= (2\langle \lambda \rangle^2 - \langle \lambda^2 \rangle) m(x) \\ p(0) &= 0 \\ p'(1) - \langle \epsilon \rangle p(1) &= -\langle \lambda \rangle n(1) - \langle \lambda \epsilon \rangle m(1). \end{aligned} \tag{3.8}$$

The solution, if it exists, to the above set of equations will yield approximations for $m(x)$, $n(x)$, and $p(x)$ which are correct through $\langle \epsilon^2 \rangle$ terms. However, there are complications in the solution of the above system as there are three unknown parameters $\langle \lambda^2 \rangle$, $\langle \epsilon \lambda \rangle$, and $\langle \lambda \rangle$ appearing in the equations. The three differential equations can be solved and the three boundary conditions at $x = 0$ used to eliminate three of the

six unknown constants. The three boundary conditions at x = 1 then yield only one "eigenvalue" equation which is a function of the three unknown parameters. Thus, we are not able to obtain a solution unless we can obtain approximations for two of the unknown parameters from another source. Whatever approximations that are used must be valid through $\langle \epsilon^2 \rangle$ terms to ensure that $\langle u \rangle (x)$ will be valid through $\langle \epsilon^2 \rangle$ terms.

One such approximation that can be used is to assume

$$\langle \lambda^2 \rangle = \langle \lambda \rangle^2 + \lambda_1^2 \ \sigma_\epsilon^2 \tag{3.9}$$

and

$$\langle \epsilon \lambda \rangle = \langle \epsilon \rangle \langle \lambda \rangle + \lambda_1 \ \sigma_\epsilon^2, \tag{3.10}$$

where $\lambda_1 = \frac{d\lambda}{d\epsilon}(0)$ and σ_ϵ^2 is the variance of ϵ. Both (3.9) and (3.10) are correct up to terms of order $\langle \epsilon^2 \rangle$ and can be derived using a perturbation series. These assumptions were used in the systems (3.6) to (3.8) to yield a very complicated eigenvalue equation for $\langle \lambda \rangle$. This latter equation was solved numerically, yielding the estimates of $\langle \lambda \rangle$ as shown in Table III. The estimates for $\langle \lambda \rangle$ from the second hierarchy can now be used in (3.9) to yield estimates for the second moment of λ. These are shown in Table IV. In both Tables we again assume ϵ uniformly distributed between 0 and α.

α	EXACT	2nd HIERARCHY	1st HIERARCHY	$\left\lvert \frac{E_2}{E_1} \right\rvert$
.5	2.9354	2.9350	2.943	.05
1	3.3460	3.3431	3.373	.11
2	4.0289	4.0091	4.116	.23
4	5.0118	4.8903	5.239	.53

TABLE III: A comparison of $\langle \lambda \rangle$ for ϵ uniformly distribued between 0 and α. $\left\lvert \frac{E_2}{E_1} \right\rvert$ is the ratio of the error for the second hierarchy to the error for the first hierarchy.

α	EXACT	2nd HIERARCHY	3 TERM PERT.	4 TERM PERT.
.5	8.6848	8.6981	8.7220	8.6803
1	11.4212	11.5096	11.6893	11.2459
2	16.8639	17.4062	18.6238	15.9571
4	26.4776	29.2483	36.4928	15.1595

TABLE IV: A comparison of the second moment of λ for ϵ uniformly distributed between 0 and α.

We see in Table III that the second hierarchy does indeed yield improved estimates for $\langle \lambda \rangle$, in fact, for small α the improvement is considerable. Unfortunately, the above method suffers from the fact that the approximations (3.9) and (3.10) are based on a perturbation series analysis and hence will not give good results for α large. This is not a typical result for hierarchy techniques, since in general perturbation series approxima-

tions are not used in conjunction with hierarchy equations. The results in Table IV are as we might have predicted, as the second hierarchy yields estimates for $\langle\lambda^2\rangle$ which are better than a three term perturbation series for all α shown. A four term perturbation series gives better estimates only for smaller α.

Before discussing an alternate method of arriving at closure assumptions for the second member of the hierarchy, it should be pointed out that the main purpose of this section was to obtain improved estimates of $\langle u\rangle(x)$. As with all eigenvalue problems, one must first find the eigenvalues, as we have done, and then use these to determine the eigenfunctions. Thus far we have obtained improved estimates for $\langle\lambda\rangle$, and hence we can use these in (3.6) through (3.10) to obtain numerical estimates for the "improved" mean value of u(x). However, due to the relatively crude nature of the assumptions (3.9) and (3.10), we do not see an improvement over the first hierarchy estimates for even small values of ϵ.

The assumptions (3.3) and (3.5) are not the only assumptions that can be made in order to close the infinite hierarchy at the second member. There is a second method for arriving at approximations similar to (3.3) to (3.5) known as the cumulant discard method [5]. Richardson uses this method to obtain improved results for initial value problems. However, with slight modifications, we are able apply the method to stochastic eigenvalue problems also. Specifically, the cumulant discard method for the n^{th} hierarchy neglects the quantity

$$\left[\frac{\partial^{n+1}}{\partial\xi_1 \ldots \partial\xi_n \partial\mu} \ln \left\langle \exp\left(i\sum_{j=1}^{n}\xi_j\beta_j + i\mu\Delta u\right)\right\rangle\right]^0 \tag{3.11}$$

where the super zero means that the expression is evaluated at $\xi_i = 0$, $i = 1, \ldots n$, and $\mu = 0$, and $\Delta u = u(x) - \langle u\rangle(x)$. The expression in (3.11) is the $(n+1)^{st}$ order cumulant corresponding to the $n + 1$ random variables $\beta_1, \ldots \beta_n$, and Δu. The cumulant discard method involves the evaluation of (3.11) and then setting it equal to zero to obtain an expression for $\langle\beta_1\beta_2 \ldots \beta_n u\rangle$ as a linear combination of the lower order moments $\langle u\rangle$, $\langle\beta_1 u\rangle$, $\langle\beta_2 u\rangle$, ..., $\langle\beta_1 \ldots \beta_{n-1} u\rangle$, with coefficients being various moments of the β_i. The modification we need to use here is that the first k $(0\leq k\leq n)$ of the β_i are now set equal to λ and the other n-k of the β_i are set equal to ϵ, thus giving us an approximation for $\langle\lambda^k\epsilon^{n-k}u\rangle$ in terms of appropriate lower order moments.

As two specific examples we will look at the cases for n = 1 and n = 2. For n = 1 (3.11) becomes

$$\left[\frac{\partial^2}{\partial\xi_1\partial\mu} \ln\langle\exp\left[i\xi_1\beta_1 + i\mu\Delta u\right]\rangle\right]^0 = \langle\beta_1\Delta u\rangle. \tag{3.12}$$

Setting (3.12) equal to zero and recalling that $\Delta u = u - \langle u\rangle$ we obtain

$$\langle\beta_1 u\rangle = \langle\beta_1\rangle\langle u\rangle. \tag{3.13}$$

From (3.13) we can obtain both the expressions $\langle\lambda u\rangle = \langle\lambda\rangle\langle u\rangle$ and $\langle\epsilon u\rangle = \langle\epsilon\rangle\langle u\rangle$ by setting $\beta_1 = \lambda$ and $\beta_1 = \epsilon$ respectively. Thus the cumulant discard method yields the same results as we obtain in section 2 for the first hierarchy.

Using the same procedure for the case n = 2 we obtain from (3.11)

$$\langle\epsilon^2 u\rangle = \langle\epsilon^2\rangle\langle u\rangle + 2\langle\epsilon\rangle\langle\epsilon u\rangle - 2\langle\epsilon\rangle^2\langle u\rangle \tag{3.14}$$

when $\beta_1 = \beta_2 = \epsilon$ and

$$\langle\lambda\epsilon u\rangle = \langle\lambda\rangle\langle\epsilon u\rangle + \langle\lambda\epsilon\rangle\langle u\rangle + \langle\epsilon\rangle\langle\lambda u\rangle - 2\langle\epsilon\rangle\langle\lambda\rangle\langle u\rangle \tag{3.15}$$

when $\beta_1 = \lambda$ and $\beta_2 = \epsilon$. When $\beta_1 = \beta_2 = \lambda$, the cumulant discard method yields the same equation as that given in (3.4). The question that should be raised now is whether the expressions (3.14) and (3.15), when used in the second member of the hierarchy, will yield better or worse results than when the simpler expressions (3.3) and (3.5) were used. This question can in part be answered by the fact that both (3.14) and (3.15) can be shown to be correct through $\langle\epsilon^2\rangle$ terms, again using perturbation series. In fact (3.14) is correct through $\langle\epsilon^3\rangle$ terms for the special case when ϵ is uniformly distributed. Thus we would again expect to obtain estimates of $\langle\lambda\rangle$ (and hence of $\langle u\rangle$) which are correct through $\langle\epsilon^2\rangle$ terms when (3.14) and (3.15) are used. In addition, when ϵ has a distribution with a first and second moment close to the first and second moment for the uniform distribution we would expect better results than those obtained previously. Unfortunately the use of (3.14) and (3.15) in the hierarchy does not alleviate the fact that the resulting eigenvalue equation contains three parameters, and hence further calculations were not carried out.

To conclude this section something needs to be said concerning the closure assumptions necessary to close the hierarchy at higher members. Theoretically the ground work is all set for using either of the methods discussed in this section. From a practical point of view, though, the computations will become so complicated that it probably would not be worthwhile to consider more members of the hierarchy, especially in view of the surprisingly good approximations obtained with just one member of the hierarchy.

4. *Higher Moments of the Eigenfunctions*

Thus far we have only attempted to find and improve estimates for the mean value of u(x). If we want to determine directly the second and higher moments for u(x), we must go back to the original equations:

$$\begin{aligned} u''(x) + \lambda u(x) &= 0 \\ u(0) &= 0 \\ u'(1) + \epsilon u(1) &= 0, \end{aligned} \tag{4.1}$$

and derive a hierarchy for the appropriate higher moments. In particular, for the second moment, multiply each equation of (4.1) by $u(x_1)$ and then find the mean value of each equation. This gives the set of equations

$$\begin{aligned} \langle u(x_1)u(x)\rangle'' + \langle\lambda u(x)u(x_1)\rangle &= 0 \\ \langle u(x_1)u(0)\rangle &= 0 \\ \left[\langle u(x_1)u(x)\rangle' + \langle\epsilon u(x_1)u(x)\rangle\right]_{x=1} &= 0, \end{aligned} \tag{4.2}$$

where the prime differentiation is differentiation with respect to x. The solution to (4.2) is found and then x_1 is set equal to x to obtain estimates for $\langle u^2\rangle(x)$. Improvements to the estimates given by (4.2) are found by writing problems for the two third order moments contained in (4.2). It appears that the same problems arise in the second member of the hierarchy for $\langle u^2\rangle(x)$ as arose in the second member of the hierarchy for $\langle u\rangle(x)$, since the only difference will be that u(x) is replaced by $u(x)u(x_1)$.

For the third and higher order moments of u(x) we multiply equations (4.1) by $u(x_1)u(x_2)$, $u(x_1)u(x_2)u(x_3)$ etc., to obtain the appropriate hierarchies. Rather than deriving these higher order hierarchies in detail, we will solve (4.2) and show that the estimate for the second moment of u(x) will be

$$\langle u^2\rangle(x) = \langle\lambda\rangle^{-1} \sin^2\langle\lambda\rangle^{1/2} \tag{4.3}$$

when the appropriate norm is used. The exact second moment is

$$\langle u^2\rangle(x) = \langle \lambda^{-1} \sin^2 \lambda^{1/2} x\rangle. \tag{4.4}$$

No numerical comparisons were made of (4.3) and (4.4), but two things can be noted about (4.3). The estimate as given in (4.3) agrees with that of (4.4) up to order $\langle\epsilon\rangle$ terms, as predicted. Secondly, if (4.3) is used to obtain estimates of the variance of u(x), it is found that the estimates will be zero for all choices of $\langle\epsilon\rangle$, which agrees with a two term perturbation series estimate, assuming only linear terms in $\langle\epsilon\rangle$ are kept.

To solve (4.2) we assume that

$$\begin{aligned} &\langle\lambda u(x)u(x_1)\rangle = \langle\lambda\rangle\langle u(x)u(x_1)\rangle \\ \text{and} \quad &\\ &\langle\epsilon u(x_1)u(1)\rangle = \langle\epsilon\rangle\langle u(x_1)u(1)\rangle, \end{aligned} \tag{4.5}$$

which are assumptions similar to the ones made in the first hierarchy for $\langle u\rangle(x)$. Equations (4.5) are also correct to terms involving $\langle\epsilon\rangle$. Using (4.5) in (4.2) we obtain

$$\langle u(x)\ u(x_1)\rangle = A(x_1)\ \sin \langle\lambda\rangle^{1/2} x, \tag{4.6}$$

where $\langle\lambda\rangle$ is given by (2.7). To determine $A(x_1)$, we can use the symmetry of x and x_1 in (4.2) to obtain

$$A(x_1) = B \sin \langle\lambda\rangle^{1/2} x_1, \tag{4.7}$$

so that

$$\langle u^2(x)\rangle = B \sin^2 \langle\lambda\rangle^{1/2} x. \tag{4.8}$$

To obtain the appropriate normalization condition we begin with

$$u'(0) = 1, \tag{4.9}$$

which is the normalization condition used for the exact solution. Equations (4.9) implies that

$$\langle (u')^2\rangle(0) = 1, \tag{4.10}$$

which is used to show that

$$\langle u^2\rangle''(0) = 2, \tag{4.11}$$

the required normalization condition arising from (4.9). The above method can be generalized to show that

$$\frac{d^n}{dx^n} \langle u^n\rangle(x) \Big|_{x=0} = n! \tag{4.12}$$

is the required normalization condition for the n^{th} moment of u(x). Using (4.11) in (4.8) we obtain the desired results (4.3). By using (4.12) in the appropriate hierarchies, it can be shown that

$$\langle u^n\rangle(x) = \langle\lambda\rangle^{-n/2} \sin^n \langle\lambda\rangle^{1/2} x, \tag{4.13}$$

which gives estimates of the n^{th} moments up to order $\langle\epsilon\rangle$ terms.

There are two alternate ways to derive estimates for the second moment of u(x). The first merely solves equations (4.2) using assumptions

other than (4.5) while the second derives directly from (4.1) a differential equation for $u^2(x)$. We will look at these methods only briefly, as they both yield the same estimate as previously found.

If the assumptions

$$\begin{aligned} &\langle \lambda u(x)u(x_1)\rangle = \langle\lambda\rangle\langle u(x)\rangle\langle u(x_1)\rangle \\ \text{and} \qquad &\langle \epsilon u(x_1)u(1)\rangle = \langle\epsilon\rangle\langle u(x_1)u(1)\rangle , \end{aligned} \tag{4.14}$$

which are correct to order $\langle\epsilon\rangle$ terms, are made in equations (4.2), we obtain the set of equations

$$\begin{aligned} &\langle u(x_1)u(0)\rangle = 0 \\ &\langle u(x_1)u(x)\rangle'' = -\langle\lambda\rangle\langle u(x)\rangle\ \langle u(x_1)\rangle \\ &\left[\langle u(x_1)u(x)\rangle'\right]_{x=1} + \langle\epsilon\rangle\langle u(x_1)u(1)\rangle = 0, \end{aligned} \tag{4.15}$$

In section 2 we found the estimate $\langle u(x)\rangle = \langle\lambda\rangle^{-1/2} \sin \langle\lambda\rangle^{1/2}\ x$, when $\langle\lambda\rangle^{1/2} = -\langle\epsilon\rangle \tan \langle\lambda\rangle^{1/2}$. Using these in (4.15), with two integrations yields the estimate (4.3) for $\langle u^2\rangle$ (x).

To obtain a problem for $u^2(x)$, we multiply equations (4.1) by u(x) to obtain

$$\begin{aligned} u(x)\ u''(x) + \lambda u^2(x) &= 0 \\ u^2(0) &= 0 \\ u(1)\ u'(1) + \epsilon u^2(1) &= 0. \end{aligned} \tag{4.16}$$

At this point we recall that

$$\begin{aligned} &(u^2)'(x) = 2u(x)u'(x) \\ \text{and} \qquad &(u^2)''(x) = 2(u')^2(x) + 2u(x)\ u''(x). \end{aligned} \tag{4.17}$$

Hence, evaluation of the first equation at x = 1 and the solution of the second for u(x) u''(x) reduces (4.16) to

$$\begin{aligned} \langle u^2\rangle''(x) + 2\langle\lambda\rangle\langle u^2\rangle(x) &= 2\langle u'\rangle^2(x) \\ \langle u^2\rangle(0) &= 0 \\ \langle u^2\rangle'(1) + 2\langle\epsilon\rangle\langle u^2\rangle(1) &= 0, \end{aligned} \tag{4.18}$$

after appropriate first order assumptions are made. If estimates of $\langle u\rangle$ (x) and $\langle\lambda\rangle$ are used from previous work then (4.18) may be solved to yield estimates for $\langle u^2\rangle$ (x), which for this example are the same as (4.3), those previously obtained.

5. *A Brief Consideration of Other Eigenvalue Problems*

Now that we have applied hierarchy equation techniques to one particular problem we would like to investigate whether similar results will be found for other typical eigenvalue problems. To do this we will look briefly at an eigenvalue problem which has its random parameter appearing in the differential equation. If we consider the vibration of a string with a random density per unit length, then we have the normal modes given by the equations

$$\begin{aligned} &u''(x) + \lambda r(x)u(x) = 0 \\ &u(0) = u(1) = 0, \end{aligned} \tag{5.1}$$

where r(x) is the random density per unit length. This is a more com-

plicated problem than that previously considered since the random variable is a function of x. For this reason we will mostly discuss the derivation of the hierarchy and various closure assumptions and not carry out a detailed comparison of the results. In general, after closure assumptions are made it is necessary to make certain assumptions about $\langle r(x)\rangle$, $\langle r(x)r(x_1)\rangle$, etc. in order to solve the resulting equations. We will not get into a detailed discussion of these, but where necessary will make use of some reasonable assumptions.

The first member for the hierarchy for the mean value of u(x) comes from (5.1) by taking the mean of each equation

$$\begin{aligned} &\langle u(x)\rangle'' + \langle \lambda r(x)u(x)\rangle = 0 \\ &\langle u(0)\rangle = \langle u(1)\rangle = 0. \end{aligned} \tag{5.2}$$

If we multiply (5.1) by $\lambda r(x_1)$ and then average, we obtain the second member of the hierarchy

$$\begin{aligned} &\langle \lambda r(x_1)u(x)\rangle'' + \langle \lambda^2 r(x_1)r(x)u(x)\rangle = 0 \\ &\langle \lambda r(x_1)u(0)\rangle = \langle \lambda r(x_1)u(1)\rangle = 0. \end{aligned} \tag{5.3}$$

The higher members are derived in a similar fashion. To close the hierarchy at the first member the simplest assumption to make concerning the third order moment $\langle \lambda r(x)u(x)\rangle$ is

$$\langle \lambda r(x)u(x)\rangle = \langle \lambda\rangle\langle r(x)\rangle\langle u(x)\rangle , \tag{5.4}$$

which, when used in (5.2), gives

$$\begin{aligned} &\langle u\rangle'' + \langle \lambda\rangle\langle r(x)\rangle\langle u\rangle = 0 \\ &\langle u\rangle (0) = \langle u\rangle (1) = 0. \end{aligned} \tag{5.5}$$

Using the appropriate perturbation series it is possible to show that (5.4) is correct through linear terms in r(x). However, we do not make any assumptions about the smallness of r(x) in using (5.4). Equations (5.5) can be solved once appropriate assumptions are made concerning $\langle r(x)\rangle$. If r(x) is assumed to be constant, then the solution to (5.5) is

$$\begin{aligned} &\langle u(x)\rangle = B \sin \pi x \\ &\langle \lambda\rangle = \frac{\pi^2}{\langle r(x)\rangle} , \end{aligned} \tag{5.6}$$

which is identical with a one term perturbation series solution under the same assumptions on $\langle r(x)\rangle$ [1].

Before going on to discuss the second member of the hierarchy, it should be pointed out that other assumptions concerning the third order moment in (5.2) can be made. They are

$$\langle \lambda r(x)u(x)\rangle = \langle \lambda r(x)\rangle\langle u(x)\rangle + \langle \lambda\rangle\langle r(x)u(x)\rangle + \langle r(x)\rangle\langle \lambda u(x)\rangle - 2\langle r(x)\rangle\langle \lambda\rangle\langle u(x)\rangle$$

and (5.7)

$$\langle \lambda r(x)u(x)\rangle = \langle \lambda r(x)\rangle\langle u(x)\rangle + \langle \lambda\rangle\langle r(x)u(x)\rangle - \langle r(x)\rangle\langle \lambda u(x)\rangle,$$

which are exactly the same assumptions as made in the second hierarchy of the previous example. From before we know that both of (5.7) are correct through second order terms involving r(x), and hence when used in (5.2) should give a better approximation than (5.6). Unfortunately, both the

assumptions in (5.7) involve the moments $\langle\lambda u(x)\rangle$ and $\langle r(x)u(x)\rangle$, and hence when they are used in (5.2) we do not obtain a closed system. It is possible to write problems for $\langle\lambda u(x)\rangle$ and $\langle r(x)u(x)\rangle$ which would then give a consistent system, but the resulting system is even more complicated than a straight-forward analysis of the second hierarchy, which will also give estimates valid through second order terms involving r(x).

The second member of the hierarchy involves the fifth order moment $\langle\lambda^2 r(x)r(x_1)u(x)\rangle$. If we assume that

$$\langle\lambda^2 r(x)r(x_1)u(x)\rangle = \langle\lambda^2\rangle\langle r(x)r(x_1)\rangle\langle u(x)\rangle \tag{5.8}$$

in (5.3), we obtain the equations

$$\begin{aligned}&\langle\lambda r(x_1)u(x)\rangle'' = -\langle\lambda^2\rangle\langle r(x)r(x_1)\rangle\langle u(x)\rangle\\&\langle\lambda r(x_1)u(0)\rangle = \langle\lambda r(x_1)u(1)\rangle = 0,\end{aligned} \tag{5.9}$$

which along with (5.2) constitute a consistent system for $\langle u(x)\rangle$, correct to second order terms in r(x). Equations (5.9) can be solved for $\langle\lambda r(x_1)u(x)\rangle$ in terms of $\langle u(x)\rangle$, giving

$$\langle\lambda r(x_1)u(x)\rangle = \langle\lambda^2\rangle\int_0^1 G(x,\eta)\langle r(\eta)r(x_1)\rangle\langle u(\eta)\rangle d\eta, \tag{5.10}$$

where $G(x,\eta)$ satisfies

$$\begin{aligned}&y'' = -\delta(x-\eta)\\&y(0) = y(1) = 0.\end{aligned} \tag{5.11}$$

Using (5.10) in (5.2) we obtain

$$\langle u(x)\rangle = \langle\lambda^2\rangle\int_0^1\int_0^1 G(x,\xi)G(\xi,\eta)\langle r(\eta)r(\xi)\rangle\langle u(\eta)\rangle d\eta d\xi \tag{5.12}$$

as the integral equation satisfied by $\langle u(x)\rangle$. Notice that its solution will give estimates not only of $\langle u(x)\rangle$, but also of $\langle\lambda^2\rangle$.

To obtain any more information concerning $\langle u(x)\rangle$ appropriate assumptions concerning $\langle r(x)r(x_1)\rangle$ must be made. For instance, if $\langle r(x)r(x_1)\rangle$ is assumed constant, then the estimes given by (5.12) reduce to those of the first hierarchy, and to the two term perturbation series solution. For more interesting assumptions about $\langle r(x)r(x_1)\rangle$, the estimates given by (5.12) will be better than a two term perturbation series estimate, as the assumption (5.8) is correct to second order terms in r(x). Again it should be mentioned that nowhere in the above work have we assumed anything about the smallness of r(x), hence the above estimates may be expected to yield results for a wide range of values of r(x). This again is a typical result for hierarchy techniques.

Other assumptions than (5.8) can be made for the fifth order moment involved in the second member of the hierarchy by using the techniques of the previous example. However, these will involve other lower order moments for which additional equations must be derived, and hence this will not be a practical method for improvements for this type of problem. Similar results hold for the higher members of the hierarchy and hence we will not go into them here.

Thus far we have looked at two distinct types of stochastic eigenvalue problems. While by no means are all stochastic eigenvalue problems of one of these types, these are many that are either of these types or else can be transformed into one of these types. Hence in conclusion we may say that hierarchy equations can be successfully applied to linear stochastic eigenvalue problems. In general very good estimates of the mo-

ments of the eigenfunction *and* the mean of the eigenvalue are obtained from the first member of the appropriate hierarchy. In theory improvements to these estimates are always possible, while in practice the calculations may become very complicated for some problems.

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